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Exact density of states for finite Gaussian random matrix ensembles via supersymmetry

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Abstract

We calculate the exact density of states (DOS) for the three classical and two non-classical random matrix ensembles for finite matrix size N using supersymmetric integrals. The $1/N$ expansion yields, already in lowest order, good approximations to the exact result even for small values of $N \sim 5$. We conjecture a connection between the N -dependence of the oscillating part of the DOS and the short-distance behaviour of the two-level correlation function.

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1. Introduction

Random matrix theory (RMT) has become a well-established tool to describe the statistical properties of the energy levels of quantum many-particle systems in the ergodic regime [1]. It is based on the assumption that (apart from global symmetry conditions as spin rotation- or time-reversal invariance) the Hamiltonian H of a system with N states is described by a probability distribution

$$\langle H_{ij} \rangle = 0 \quad \langle H_{ij} H_{kl}^* \rangle = g \delta_{ik} \delta_{jl}.$$

RMT derives the statistical properties of the eigenvalues of H from this distribution of its matrix elements. Whereas most of the applications are still centred around the three classical Wigner–Dyson (WD) ensembles, the Gaussian unitary, orthogonal and symplectic ensembles (GUE, GOE and GSE), there is also interest in certain generalizations of the standard cases, leading to seven additional symmetry classes, which differ from each other by their behaviour under a set of discrete spacetime transformations. Three of them are the chiral analogues to the WD ensembles and four are relevant for mesoscopic normal-superconducting hybrid systems [2].

RMT is a soluble description of the statistics of disordered and chaotic systems in the sense that all n -level correlation functions can be computed in principle exactly. The most important of these correlation functions is the two-level correlation function as it already

contains the main information about the repulsion of neighbouring levels and discerns the three WD ensembles unambiguously. Several mathematical tools are available to compute these correlation functions, e.g. the method of orthogonal polynomials [3] or the supersymmetric method [5]. The last method maps the matrix ensemble to a zero-dimensional nonlinear sigma model in the limit of large matrix size N . Because then the limit $N \rightarrow \infty$ is implicit in all calculations, it is not possible to study directly finite-size effects, i.e. the dependence of the average level spacing $\Delta(E)$ at energy E on N .

Instead one is forced to fix a certain scaling of the overall normalization of H with N . In this way the two energy scales, the bandwidth $2E_{\max}$ and the mean level spacing $\Delta(E)$, are entangled with the matrix size. Either the bandwidth is kept at a fixed finite value (macroscopic scaling) or the mean level spacing $\Delta(E) \sim 1$ at a certain energy E (microscopic scaling). The first scaling is used for the DOS (and yields a semicircle for $N \rightarrow \infty$), the second is appropriate to extract the universal level repulsion. The non-classical ensembles (we will treat classes D and C in this paper) show characteristic features of the DOS, deviating from the classical ensembles, but this was calculated only in the microscopic scaling limit [2, 6, 7]. The DOS varies strongly on the scale of level spacing for $E \sim 0$, and therefore becomes dependent on the position within the spectrum. In this case the usual unfolding procedure which normalizes the DOS to a constant over many consecutive levels is inapplicable. The question arises, whether the new features of the DOS survive the macroscopic scaling $\Delta(E) \sim 1/N$ leading to a finite bandwidth, which is the physically relevant case for solid-state applications. It seems, therefore, advisable to compute the DOS exactly for finite N and to study its behaviour in the limit $N \rightarrow \infty$ using this formula. The finite- N case is interesting in its own right (see, e.g., [8]).

Results for finite N were obtained via the orthogonal polynomial method [3], where the exact DOS for finite N is given for the three WD ensembles. Additionally, Itoi *et al* have calculated the smooth part of the correction for the WD ensembles using different methods [4]. They missed, however, the important oscillatory part (see below), and assumed integer exponents in the $1/N$ expansion which is not true for the GSE. We will show in the following that the supersymmetric method can also be used and is technically simpler. The possibility of calculating the DOS for finite N was briefly mentioned in [9].

The outline of the paper is as follows. In section 2, we study the simplest case, the GUE, and explain in detail how the exact DOS for finite N can be obtained. We then use a saddle-point method to derive a $1/N$ expansion. This expansion is not based on a supersymmetric saddle-point manifold but, on the contrary, is effected by exact integration over the fermionic variables, which inevitably breaks explicit supersymmetry. The first nonvanishing term in this expansion already yields excellent approximations to the exact result for small N and everywhere in the spectrum except at the band edge, where this correction diverges, due to the invalidity of the chosen saddle point. In section 3, we give the exact DOS for finite N of the orthogonal and symplectic ensembles as well as classes C and D. These ensembles are especially simple among the non-classical ones because the joint eigenvalue distribution can be associated with free fermions on a line in an external potential [2]. Nevertheless, the calculations of the DOS become simple only if the confining part of the potential is neglected, corresponding to microscopic scaling, thereby losing all information about the relation between $\Delta(E)$ and N which is just given by the confinement. Therefore here we treat class C/D together with the WD ensembles. Regarding the level repulsion they belong to the unitary universality class, i.e. without time reversal invariance (see below). Our method, of course, extends to all other Gaussian ensembles as well. Section 4 contains the $1/N$ expansion around the bosonic saddle points for GOE, GSE and class C/D. We observe a relation between the N -dependence of the oscillatory part of the correction terms and the short-distance behaviour of the two-level

correlation function. Then we determine the average level spacing at the band centre, where the DOS of classes C and D deviates strongly from the (constant) behaviour known in the WD ensembles, in a way independent of any scaling assumption. Section 5 contains a summary of our results.

2. The unitary ensemble

The ensemble consists of complex Hermitian $N \times N$ -matrices $H = H^\dagger$. We represent the DOS at complex energy z ($\text{Im } z > 0$) as the following expectation value (summation convention is understood):

$$\langle n(z) \rangle = \frac{1}{\pi N} \text{Re} \int \mathcal{D}[\phi, \phi^\dagger, H] x_i^* x_i e^{-S_H}. \tag{1}$$

Here $\phi = (x_i, \psi_i)^T$, $\phi^\dagger = (x_i^*, \bar{\psi}_i)$ is a vector with N bosonic and N fermionic components. We set the variance of H to $g = 1/2$. The action reads

$$S_H = i\phi_i^\dagger (H_{ij} - z\delta_{ij})\phi_j + \text{tr} H^2. \tag{2}$$

Integration over H yields an effective action

$$S_4 = -iz\phi_i^\dagger \phi_i + \frac{1}{4} \text{trg}(\phi_i \phi_i^\dagger)^2. \tag{3}$$

A Hubbard–Stratonovich transformation to \mathbf{Q} variables [10] (the q, p are bosonic and the $\vartheta, \bar{\vartheta}$ are fermionic),

$$\mathbf{Q} = \begin{pmatrix} q & \bar{\vartheta} \\ \vartheta & ip \end{pmatrix} \tag{4}$$

gives an action bilinear in ϕ :

$$S_{\mathbf{Q}} = i\phi_i^\dagger (\mathbf{Q} - z)\phi_i + \text{trg} \mathbf{Q}^2. \tag{5}$$

Now we integrate over the ϕ fields *and* the Grassmann variables $\vartheta, \bar{\vartheta}$ exactly to get an action, which depends only on the two real variables q and p . The DOS reads

$$\begin{aligned} \langle n(z) \rangle &= \frac{-2}{\pi N} \text{Im} \int \mathcal{D}[\mathbf{Q}] \frac{q e^{-\text{trg} \mathbf{Q}^2}}{\det g(\mathbf{Q} - z)^N} = \frac{-2}{\pi N} \text{Im} \int \frac{dp dq}{\pi} q e^{-p^2 - q^2} \frac{(ip - z)^N}{(q - z)^N} \\ &\times \left(1 - \frac{N}{2(q - z)(ip - z)} \right). \end{aligned} \tag{6}$$

If we now set $z = E + i\epsilon$ with $E \in \mathbb{R}$ and perform the integration over p and q , we get the exact finite- N result [3]

$$\langle n(E) \rangle = \frac{e^{-E^2}}{2^N N! \sqrt{\pi}} (H_N^2(E) - H_{N+1}(E)H_{N-1}(E)) \tag{7}$$

where $H_n(E)$ denotes the n th Hermite polynomial. To perform an expansion around the limit $N = \infty$, we introduce a rescaled energy variable $x = E/\sqrt{N}$. With this scaling the bandwidth becomes finite, and the DOS a continuous normalizable distribution in the limit $N \rightarrow \infty$. After the rescaling (which is done for p and q as well) we get for the DOS (after integration over $\vartheta, \bar{\vartheta}$ but before integration over p and q)

$$\langle n(x) \rangle = \frac{-2}{\pi} \text{Im} \int \frac{dp dq}{\pi/N} q e^{-Np^2 - Nq^2} \frac{(ip - x)^N}{(q - x)^N} \left(1 - \frac{1}{2(q - x)(ip - x)} \right) \tag{8}$$

which can be written as

$$\langle n(x) \rangle = \int dp dq e^{-NS(p,q,x)} f(p, q, x). \tag{9}$$

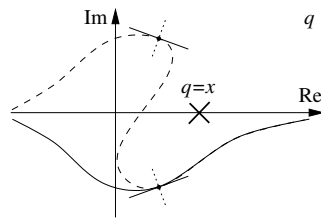


Figure 1. The saddle points of q and possible integration paths. Full lines denote the stable directions and dotted lines the unstable directions.

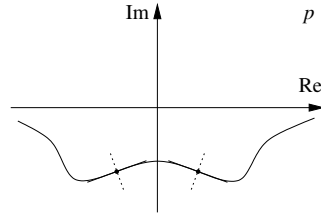


Figure 2. The saddle points for p .

The large prefactor N in the exponent allows for a saddle-point approximation of the p , q integrals in the usual way. We get four solutions of the saddle-point equations

$$q_{\pm} = x/2 \pm i/2\sqrt{2-x^2} \quad p_{\pm} = -iq_{\pm}. \tag{10}$$

The stability matrix is

$$\frac{\partial^2 S}{2\partial q^2} = 1 - q_{\pm}^2 = \pm i\sqrt{2-x^2}q_{\pm} =: \Delta_{\pm} \tag{11}$$

$$\frac{\partial^2 S}{2\partial p^2} = \Delta_{\pm} \tag{12}$$

$$\frac{\partial^2 S}{\partial p\partial q} = 0. \tag{13}$$

Figures 1 and 2 show the location of the saddle points in the complex plane along with their stable directions and possible integration paths. Integral (6) converges only for $\text{Im } z > 0$. This means that the integration path of q must pass below the singularity at $q = x$. This forces us to use only the saddle point q_- of the q integration: a path of integration which also crosses q_+ leads through the region between the two saddle points, where the integrand of (8) is large. Therefore, the correct choice for the path of integration over q is the full line in figure 1. In contrast, it is possible to use an integration path which goes through both saddle points for the p integral, which is shown in figure 2. After the change of variables

$$q \rightarrow q_- + \frac{\delta q}{\sqrt{N\Delta_-}} \quad p \rightarrow p_{\pm} + \frac{\delta p}{\sqrt{N\Delta_{\pm}}} \tag{14}$$

we may perform an expansion in δq and δp , which defines the $1/N$ expansion. The action $S_{\text{SP}}^{(1)} = S(q_-, p_-)$ vanishes and yields the well-known semicircle law as the only contribution surviving in the $N \rightarrow \infty$ limit. The action $S_{\text{SP}}^{(2)} = S(q_-, p_+)$ does not vanish, but is purely

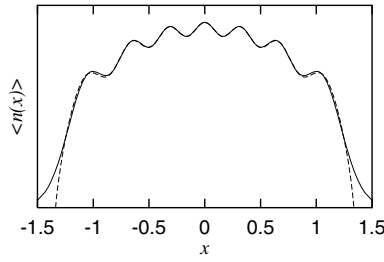


Figure 3. Exact DOS of the GUE for $N = 7$. The dashed line is the $1/N$ approximation.

imaginary and yields a contribution $\sim e^{-N S_{\text{SP}}^{(2)}}$ of modulus 1. This contribution disappears in the $N = \infty$ limit, because the factor $(1 - \frac{1}{2(q-x)(ip-x)})$ vanishes for $q = q_-, p = p_+$. It yields, however, an important oscillatory contribution $\propto 1/N$, whereas the (non-oscillatory) corrections to the first saddle point (q_-, p_-) start with order $1/N^2$. The final result is

$$\langle n(x) \rangle = \frac{\sqrt{2-x^2}}{\pi} - \frac{(-1)^N \cos [N S_{\text{SP}}^{(2)}(x)]}{\pi \sqrt{2N}(2-x^2)} + O\left(\frac{1}{N^2}\right) \tag{15}$$

with

$$S_{\text{SP}}^{(2)}(x) = x\sqrt{2-x^2} + 2 \arcsin(x/\sqrt{2}) \tag{16}$$

and

$$\frac{d}{dx} S_{\text{SP}}^{(2)}(x) = 2\sqrt{2-x^2} = 2\pi n_0(x) \tag{17}$$

where the zeroth order approximation to the DOS (the semicircle) is written as $n_0(x) = (1/\pi)\sqrt{2-x^2}$. The exact density of states and the $1/N$ approximation are shown in figure 3. From (16) and (17) we note that the number of maxima of the oscillatory $1/N$ term equals N and the local width between two maxima scales as the inverse of the DOS in zeroth approximation. The positions of the maxima give, therefore, the locations of eigenvalues for a ‘typical’ realization of a random matrix from the GUE with one state per maximum on average. Identifying the distance between two adjacent maxima with the average level spacing at $x = E/\sqrt{N}$, for large N we get the following expression for $\Delta(E)$:

$$\Delta(E) \approx \frac{1}{\sqrt{N} n_0(E/\sqrt{N})} \tag{18}$$

which is valid for E not too close to the band edge. The $1/N$ correction obviously diverges at the band edge, because the saddle points coalesce and as a consequence the stability matrix vanishes.

3. The other ensembles

The elements of the orthogonal, symplectic and class C/D ensembles are conveniently represented as $2N \times 2N$ -matrices with a 2×2 -block structure:

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{19}$$

where the $N \times N$ -matrices A, B, C, D fulfill a set of conditions defining different ensembles. We set the variance of H to $g = 1$ in the following, which yields a bandwidth of 4 in all cases.

3.1. The orthogonal ensemble

The elements of the GOE are real symmetric $2N \times 2N$ -matrices: $H = H^T = H^*$. To simplify calculations, it is convenient to use instead of H the unitary equivalent matrix $M = UHU^\dagger$ with $U = \exp(i(\pi/4)\sigma_1) \otimes \mathbb{1}_N$, where σ_1 is a Pauli matrix acting on the 2×2 -blocks. If H is real symmetric, M fulfils the relation $M = \sigma_1 M^T \sigma_1$. This entails the following relation among the block matrices

$$M = \begin{pmatrix} A & B \\ B^\dagger & A^T \end{pmatrix} \tag{20}$$

where A is a Hermitian $N \times N$ -matrix and B is complex symmetric. As in section 2, we write for the DOS

$$\langle n(z) \rangle = \frac{1}{2\pi N} \operatorname{Re} \int \mathcal{D}[\phi, \phi^\dagger, M](x_{1i}^* x_{1i}) e^{-S} \tag{21}$$

with $\phi = (x_1, x_2, \psi_1, \psi_2)^T$, where the $x_{1/2}, \psi_{1/2}$ are N -dimensional complex bosonic, respectively fermionic vectors and S reads

$$S = i\phi^\dagger(M - z)\phi + \frac{1}{2} \operatorname{tr} M^2. \tag{22}$$

Now we introduce the notation

$$\chi_{1j} = (x_{1j}, x_{2j}^*, \psi_{1j}, \bar{\psi}_{2j})^T \quad \chi_{2j} = (x_{2j}, x_{1j}^*, \psi_{2j}, \bar{\psi}_{1j})^T \tag{23}$$

(j is the index of the N -dimensional tensor component) together with the transposition

$$\chi_{1j}^t = (x_{1j}^*, x_{2j}, \psi_{1j}^*, -\psi_{2j}) \quad \chi_{2j}^t = (x_{2j}^*, x_{1j}, \psi_{2j}^*, -\psi_{1j}). \tag{24}$$

After integration over M , we get the quartic action in χ, χ^t :

$$S_4 = -i\chi_{1i}^t \chi_{1i} + \frac{1}{8} \operatorname{trg}(\chi_{1i} \chi_{1i}^t + \chi_{2i} \chi_{2i}^t)^2. \tag{25}$$

After transformation to the \mathbf{Q} matrix

$$\mathbf{Q} = \begin{pmatrix} q_1 & q_2^* & \bar{\vartheta}_1 & -\vartheta_2 \\ q_2 & q_1 & \bar{\vartheta}_2 & -\vartheta_1 \\ \vartheta_1 & \vartheta_2 & ip & 0 \\ \bar{\vartheta}_2 & \bar{\vartheta}_1 & 0 & ip \end{pmatrix} \tag{26}$$

we obtain

$$\langle n(z) \rangle = \frac{-1}{\pi N} \operatorname{Im} \int \mathcal{D}[\mathbf{Q}] \frac{q_1 e^{-\frac{1}{2} \operatorname{trg} \mathbf{Q}^2}}{\operatorname{detg}(\mathbf{Q} - z)^N} \tag{27}$$

with

$$\operatorname{detg} \mathbf{Q} = \frac{Q}{(ip)^2} \exp\left(-\frac{2}{Q(ip)}(q_1 \Theta_1 - q_2 \Theta_2 - q_2^* \bar{\Theta}_2) - \frac{2\Theta_2 \bar{\Theta}_2}{Q(ip)^2}\right) \tag{28}$$

and $Q = q_1^2 - q_2 q_2^*$, $\Theta_1 = \bar{\vartheta}_1 \vartheta_1 + \bar{\vartheta}_2 \vartheta_2$, $\Theta_2 = \bar{\vartheta}_1 \vartheta_2$. Note that the Grassmann integration contains now quartic terms in $\vartheta, \bar{\vartheta}$, which can nevertheless be done exactly. The result is ($q = q_1, r = q_2 q_2^*$)

$$\langle n(z) \rangle = \frac{-1}{\pi N} \operatorname{Im} \int \frac{dp dq}{\pi} dr e^{-p^2 - q^2 - r} q \frac{(ip - z)^{2N}}{[(q - z)^2 - r]^N} \times \left(1 - \frac{2N(q - z)}{[(q - z)^2 - r](ip - z)} + \frac{(2N - 1)N}{2[(q - z)^2 - r](ip - z)^2}\right). \tag{29}$$

Setting now as above $z = E + i\epsilon$ and integrating over p, q and r , we obtain the exact DOS in terms of Hermite polynomials and the error function:

$$\langle n(E) \rangle = \frac{e^{-E^2}}{4^N N \sqrt{\pi}} \left(\frac{H_{2N-1}(E)^2 - H_{2N}(E)H_{2N-2}(E)}{(2N-2)!} - \frac{H_{2N-1}(E)}{(N-1)!} \times \left(\sum_{k=1}^{N-1} \frac{2k!}{(2k)!} H_{2k-1}(E) - \sqrt{\frac{\pi}{2}} e^{\frac{E^2}{2}} \operatorname{erf} \frac{E}{\sqrt{2}} \right) \right). \tag{30}$$

3.2. The symplectic ensemble

Here we have

$$H = \begin{pmatrix} A & B \\ B^\dagger & A^T \end{pmatrix} \tag{31}$$

with A is Hermitian and B is complex antisymmetric. Proceeding as in the case of the GOE (the definition of χ, χ' is the same), we find the quartic action

$$S_4 = \frac{1}{8} \operatorname{trg}(\chi_{1i} \chi'_{1i} + \Sigma_3 \chi_{2i} \chi'_{2i} \Sigma_3)^2 \tag{32}$$

where $\Sigma_3 = \sigma_3 \otimes \mathbf{1}_N$. The corresponding \mathbf{Q} matrix reads now

$$\mathbf{Q} = \begin{pmatrix} q & 0 & \bar{\vartheta}_1 & \bar{\vartheta}_2 \\ 0 & q & -\vartheta_2 & -\vartheta_1 \\ \vartheta_1 & \bar{\vartheta}_2 & ip_1 & ip_2^* \\ \vartheta_2 & \bar{\vartheta}_1 & ip_2 & ip_1 \end{pmatrix} \tag{33}$$

and the DOS reads

$$\langle n(z) \rangle = \frac{-1}{\pi N} \operatorname{Im} \int \mathcal{D}[\mathbf{Q}] \frac{q e^{-\frac{1}{2} \operatorname{trg} \mathbf{Q}^2}}{\detg(\mathbf{Q} - z)^N} \tag{34}$$

with

$$\detg \mathbf{Q}^{-1} = \frac{P}{q^2} \exp \left(\frac{2}{Pq} (ip_1 \Theta_1 - ip_2 \Theta_2 - ip_2^* \bar{\Theta}_2) - \frac{2\Theta_2 \bar{\Theta}_2}{Pq^2} \right) \tag{35}$$

and $P = (ip_1)^2 - (ip_2)(ip_2^*)$, $\Theta_1 = \bar{\vartheta}_1 \vartheta_1 + \bar{\vartheta}_2 \vartheta_2$ and $\Theta_2 = \bar{\vartheta}_2 \vartheta_1$. Setting $p = p_1$ and $r = p_2 p_2^*$, equation (34) reads after integration over the Grassmann variables,

$$\langle n(z) \rangle = \frac{-1}{\pi N} \operatorname{Im} \int \frac{dp dq}{\pi} dr q e^{-p^2 - q^2 - r} \frac{[(ip - z)^2 + r]^N}{(q - z)^{2N}} \times \left(1 - \frac{2N(ip - z)}{[(ip - z)^2 + r](q - z)} + \frac{(2N + 1)N}{2[(ip - z)^2 + r](q - z)^2} \right) \tag{36}$$

and

$$\langle n(E) \rangle = \frac{e^{-E^2}}{(2N)! \sqrt{\pi}} \left(\frac{1}{4^N} (H_{2N}(E)^2 - H_{2N+1}(E)H_{2N-1}(E)) - \frac{N!}{2N} H_{2N}(E) \sum_{k=0}^{N-1} \frac{H_{2k}(E)}{4^k k!} \right). \tag{37}$$

3.3. Class D

The elements of class D are parametrized as

$$H = \begin{pmatrix} A & B \\ B^\dagger & -A^T \end{pmatrix} \tag{38}$$

with A Hermitian and B complex antisymmetric. The quartic action reads (in the notation above)

$$S_4 = \frac{1}{8} \text{trg}((\chi_{1i} \chi_{1i}^\dagger + \chi_{2i} \chi_{2i}^\dagger) \Sigma_3)^2 \tag{39}$$

and the \mathbf{Q} matrix

$$\mathbf{Q} = \begin{pmatrix} q_1 & q_2^* & -\vartheta_1 & -\vartheta_2 \\ q_2 & -q_1 & \bar{\vartheta}_2 & \bar{\vartheta}_1 \\ \bar{\vartheta}_1 & \vartheta_2 & ip & 0 \\ \bar{\vartheta}_2 & \vartheta_1 & 0 & -ip \end{pmatrix}. \tag{40}$$

For $\langle n(z) \rangle$ we get

$$\langle n(z) \rangle = \frac{-1}{\pi N} \text{Im} \int \mathcal{D}[\mathbf{Q}] \frac{q_1 e^{-\frac{1}{2} \text{trg} \mathbf{Q}^2}}{\text{detg}(\Sigma_3 \mathbf{Q} - z)^N} \tag{41}$$

with

$$\text{detg} \Sigma_3 \mathbf{Q} = \frac{Q}{(ip)^2} \exp \left(-\frac{2}{Q(ip)} (q_1 \Theta_1 - q_2 \Theta_2 + q_2^* \bar{\Theta}_2) - \frac{2\Theta_2 \bar{\Theta}_2}{Q(ip)^2} \right) \tag{42}$$

and $Q = q_1^2 + q_2 q_2^*$, $\Theta_1 = \bar{\vartheta}_1 \vartheta_1 - \bar{\vartheta}_2 \vartheta_2$, $\Theta_2 = \vartheta_1 \vartheta_2$. The integrand of (41) contains a pole depending on $q = q_1$ and $r = q_2 q_2^*$. To circumvent it, we have to deform the integration path into the complex plane

$$q \rightarrow e^{-i\pi/4} q + \epsilon e^{i\pi/4} \tanh(q/\epsilon) \tag{43}$$

$$\sqrt{r} \rightarrow e^{i\pi/4} \sqrt{r} + \epsilon e^{-i\pi/4} \tanh(\sqrt{r}/\epsilon) \tag{44}$$

for $\text{Re } z > 0$ and ϵ must be small enough: $0 < \epsilon < (\text{Im } z + \text{Re } z)/\sqrt{2}$. We then find

$$\langle n(z) \rangle = \frac{-1}{\pi N} \text{Im} \int \frac{dp dq}{\pi} dr q e^{-p^2 - q^2 - r} \frac{(ip - z)^{2N}}{[(q - z)^2 + r]^N} \left(1 - \frac{N(2N - 1)}{2[(q - z)^2 + r](ip - z)^2} \right) \tag{45}$$

and

$$\begin{aligned} \langle n(E) \rangle &= \frac{-(-1)^N e^{-E^2}}{N! 4^N \sqrt{\pi}} \left(\frac{(-1)^N (N - 1)!}{(2N - 2)!} H_{2N}(E) H_{2N-2}(E) + 2E H_{2N-1}(E) \right. \\ &\quad \left. \times \left(\sum_{k=1}^{N-1} \frac{(-1)^k k!}{(2k)!} H_{2k}(E) + 1 + \frac{1}{2E^2} \right) \right). \end{aligned} \tag{46}$$

3.4. Class C

The parametrization reads

$$H = \begin{pmatrix} A & B \\ B^\dagger & -A^T \end{pmatrix} \tag{47}$$

Table 1. The additional saddle points for the GOE and GSE.

Ensemble	p	q	r
GOE	$-i\left(\frac{x}{2} \pm i\sqrt{1 - \frac{x^2}{4}}\right)$	$\frac{x}{2}$	$\frac{x^2}{4} - 1$
GSE	$-i\frac{x}{2}$	$\frac{x}{2} \pm i\sqrt{1 - \frac{x^2}{4}}$	$1 - \frac{x^2}{4}$

with A Hermitian and B complex symmetric. The quartic action is

$$S_4 = \text{trg}(\chi_{1i} \chi_{1i}^\dagger \Sigma_3 + \Sigma_3 \chi_{2i} \chi_{2i}^\dagger)^2 \tag{48}$$

with the \mathbf{Q} matrix

$$\begin{pmatrix} q & 0 & -\vartheta_1 & \bar{\vartheta}_2 \\ 0 & -q & -\vartheta_2 & \bar{\vartheta}_1 \\ \bar{\vartheta}_1 & \bar{\vartheta}_2 & ip_1 & ip_2^* \\ \vartheta_2 & \vartheta_1 & ip_2 & -ip_1 \end{pmatrix}. \tag{49}$$

We have

$$\langle n(z) \rangle = \frac{-1}{\pi N} \text{Im} \int \mathcal{D}[\mathbf{Q}] \frac{q_1 e^{-\frac{1}{2} \text{trg} \mathbf{Q}^2}}{\text{detg}(\Sigma_3 \mathbf{Q} - z)^N} \tag{50}$$

with

$$\text{detg}(\Sigma_3 \mathbf{Q})^{-1} = \frac{P}{q^2} \exp\left(\frac{2}{Pq}(ip_1 \Theta_1 + ip_2 \Theta_2 - ip_2^* \bar{\Theta}_2) - \frac{2\Theta_2 \bar{\Theta}_2}{Pq^2}\right) \tag{51}$$

and $P = (ip_1)^2 + (ip_2)(ip_2^*)$, $\Theta_1 = \bar{\vartheta}_1 \vartheta_1 - \bar{\vartheta}_2 \vartheta_2$ and $\Theta_2 = \bar{\vartheta}_1 \bar{\vartheta}_2$. With $p = p_1$, $r = p_2 p_2^*$ the result reads

$$\langle n(z) \rangle = \frac{-1}{\pi N} \text{Im} \int \frac{dp dq}{\pi} dr e^{p^2 - q^2 - r} q \frac{[(ip - z)^2 - r]^N}{(q - z)^{2N}} \left(1 - \frac{N(2N + 1)}{2[(ip - z)^2 - r](q - z)^2}\right) \tag{52}$$

$$\langle n(E) \rangle = \frac{(-1)^N N! e^{-E^2}}{(2N)! \sqrt{\pi}} \left(H_{2N}(E) \sum_{k=0}^N \frac{(-1)^k H_{2k}(E)}{k! 4^k} + \frac{H_{2N+2}(E)}{4N} \sum_{k=0}^{N-1} \frac{(-1)^k H_{2k}(E)}{k! 4^k} \right). \tag{53}$$

4. Saddle-point approximation

The saddle-point approximation for the four ensembles proceeds as in the unitary case, in general: all four ensembles share the saddle points $q_{\pm} = x/2 \pm i\sqrt{1 - x^2/4}$, $p_{\pm} = -iq_{\pm}$, $r_0 = 0$ ($x = E/\sqrt{N}$).¹ As in the unitary case, the points (q_+, p_{\pm}, r_0) have to be avoided by the integration path. The fluctuations are quadratic in p and q and linear in r around the two remaining SP (q_-, p_{\pm}, r_0) . Apart from them, we have additional SP for the GOE and the GSE, given in table 1.

¹ Strictly speaking, r is at the boundary of the integration domain and therefore not a saddle point, but nevertheless corresponds to an extremum of the action.

The additional SP cannot be reached in the GOE case ($r_{\text{SP}} < 0$), but has to be included for the GSE ($r_{\text{SP}} > 0$). In classes C and D there is an additional saddle point manifold at $x = 0$. However, our results displayed below are obtained by confining the analysis to the standard saddle points, which exist for all values of x , thereby assuming continuity in the limit $x \rightarrow 0$. In principle, this procedure could have led to an additional singularity at $x = 0$ similar to the divergence at the band edge. But the comparison with the exact DOS in these cases shows that our assumption is indeed correct and the additional saddle points play no role in computing the $1/N$ correction. This view is supported by the work of Ivanov [7], who calculates the saddle-point manifold at $x = 0$ to derive the DOS in the microscopic scaling.

We give in the following the results for the $1/N$ expansion of the GOE, GSE and class C/D.

4.1. The orthogonal ensemble

Let us define the $N = \infty$ approximation to the DOS of the GOE as $n_0(x) = (1/\pi)\sqrt{1 - x^2/4}$. Then the $1/N$ expansion reads to order $1/N^2$

$$\begin{aligned} \langle n(x) \rangle = n_0(x) &- \frac{1}{N} \frac{1}{8\pi^2 n_0(x)} + \frac{1}{N^2} \frac{3 + x^2}{128\pi^6 n_0(x)^5} \\ &- \frac{1}{N^2} \frac{1}{64\pi^6 n_0(x)^5} \cos[2NS_0(x) - \arcsin(x/2)] \end{aligned} \quad (54)$$

with

$$S_0(x) = 2 \arcsin(x/2) + x\sqrt{1 - x^2/4} \quad (55)$$

$$S'_0(x) = 2\pi n_0(x) \quad (56)$$

analogous to the unitary case. We observe the same features as in the GUE case. The number of maxima of the oscillatory part within the band $-2 < x < 2$ given by the semicircle is $2N$, the total number of levels, so we have one level per maximum. The correction diverges close to the band edge as expected. Apart from the oscillatory contribution there is a non-oscillatory contribution of order $1/N$. However, the information about the level repulsion is encoded in the oscillating part $\propto \cos(2NS_0(x))$. This term is of order $1/N^2$, which means that the repulsion of levels is weaker in the GOE than in the GUE. Formally, we can write

$$\langle n(x) \rangle = n_0(x) + \frac{1}{N}(\text{non-oscillating term}) + \frac{1}{N^\alpha}(\text{oscillating term}) \quad (57)$$

with $\alpha = 2$. For the GUE $\alpha = 1$. Now the two-level correlation function $R_2(r)$ behaves for the Wigner–Dyson ensembles as [3]

$$R_2(r) \sim r^\beta \quad \text{for } r \ll 1 \quad (58)$$

where r is the distance between two levels on a scale corresponding to the average level spacing $\Delta(x) \sim 1$. For the GUE we have $\beta = 2$ and for the GOE $\beta = 1$. We conjecture, therefore,

$$\beta = \frac{2}{\alpha} \quad (59)$$

as a relation between the universal parameter β , which characterizes the short-distance behaviour of the two-level correlation function, and the exponent α of the factor $1/N$, which multiplies the oscillating term in the $1/N$ expansion. With this relation we can extract from the $1/N$ expansion of the one-point function universal information about the two-point function, which discerns the WD ensembles from each other (whereas the DOS is the same for all three ensembles in the limit $N = \infty$).

4.2. The symplectic ensemble

We have (with the same definition of $n_0(x)$ and $S_0(x)$ as in the GOE)

$$\langle n(x) \rangle = n_0(x) + \frac{1}{N} \frac{1}{8\pi^2 n_0(x)} - \frac{1}{N^{\frac{1}{2}}} \frac{\cos \left[NS_0(x) + \frac{1}{2} \arcsin(x/2) \right]}{(-1)^N \sqrt{8\pi^{\frac{3}{4}} n_0(x)^{\frac{1}{4}}}}. \tag{60}$$

Because each eigenvalue appears twice in a $2N \times 2N$ -matrix from the symplectic ensemble due to Kramers' degeneracy [3], we have only N different eigenvalues and therefore N maxima within the band. The oscillatory term comes with a $1/\sqrt{N}$ prefactor, therefore $\alpha = 1/2$, which gives with (59) the correct universal exponent $\beta = 4$.

4.3. Class C

The $1/N$ expansion yields

$$\langle n(x) \rangle = n_0(x) + \frac{1}{N} \frac{1}{8\pi^2 n_0(x)} - \frac{1}{N} \frac{\sin[2NS_0(x) + \arcsin(x/2)]}{4\pi^3 x n_0(x)^2}. \tag{61}$$

Each element of class C ensemble has a spectrum symmetric with respect to $x = 0$. To each eigenvalue x there is a state with eigenvalue $-x$. The number of maxima in the band equals $2N$, we have again one (non-degenerate) level per maximum. Now the spectral region around $x = 0$ is different from the WD ensembles because there is no smooth limit of $\langle n(x) \rangle$ for $N \rightarrow \infty$. If one introduces the rescaled variable $y = 2Nx/\pi$ and considers the microscopic limit $x \rightarrow 0, N \rightarrow \infty$, keeping y finite, the DOS reads

$$\langle n(y) \rangle = \frac{1}{\pi} \left(1 - \frac{\sin(2\pi y)}{2\pi y} \right) \tag{62}$$

which coincides with the result in [2]. In terms of y , the correction term is independent of N in contrast to the GUE, where the oscillatory contribution to the DOS vanishes as $1/N$ for large N . This, however, is not a signal of an enhanced level repulsion close to $x = 0$ in class C. It is merely due to the lack of (approximate) translational invariance at the band centre, which is caused by the reflection symmetry mentioned above. In the GUE, translational invariance is broken only through the term proportional to $\exp(-\sum_{i=1}^N \lambda_i^2)$ in the joint probability distribution of the eigenvalues λ_i , which can be neglected close to the band centre and large N . Therefore, the oscillatory structure in $\langle n(x) \rangle$ is smeared out by the summation over all members of the ensemble, whereas in class C the quotient of the probability to find a level at the first maximum with respect to the first minimum of $\langle n \rangle$ is independent of N . The spectrum is therefore more rigid near $x = 0$. The two-level correlation function $R_2(y_1, y_1 + y)$ nevertheless behaves for $0 < y_1, y \ll 1$ as y^2 and belongs therefore to the unitary universality class [2]. The correction term does not modify the macroscopic behaviour of $\langle n \rangle$ because the function $\sin(Nx)/(Nx)$ tends to zero in the L_2 sense for $N \rightarrow \infty$. This means that the probability of finding m states in a region of width $\Delta x = \frac{\pi m}{2N}$ around zero tends to one for $1 \ll m \ll N$ in the limit $N \rightarrow \infty$, as in the WD ensembles.

Because the $1/N$ approximation is reliable everywhere except at the band edge, we can compute the average level spacing exactly in the vicinity of $x = 0$ without recourse to measure x (respective y) in units of the mean spacing at a distance of many spacings from zero [2]. In terms of y (which is *exactly* related to x and the original variable E through N), the average spacing between the first and second level >0 is given by

$$\langle \Delta_y(1, 2) \rangle = \frac{1}{2\pi} (z_4 - z_2) \approx 1 \tag{63}$$

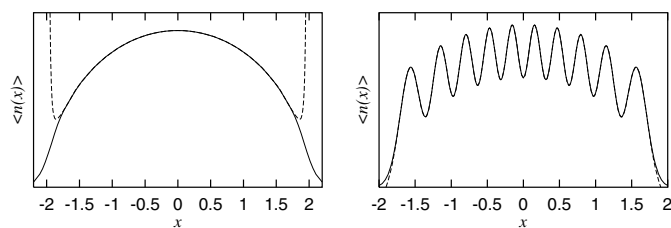


Figure 4. The GOE and the GSE. $N = 10$ in both cases. The full line is the exact DOS, the dashed line is the saddle-point approximation.

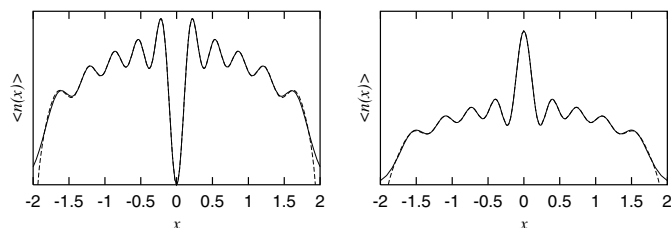


Figure 5. Classes C and D. $N = 5$ in both cases. The full line is the exact DOS, the dashed line is the saddle-point approximation.

where $z_k > 0$ denotes the k th zero of $\tan(z) - z$. This relation is valid for large N but $\Delta(j, j+1)$ can be computed via (61) for all levels j and $j+1$ not too close to the band edge and arbitrary N .

4.4. Class D

The same reflection symmetry as in class C is valid in class D. We have

$$\langle n(x) \rangle = n_0(x) - \frac{1}{N} \frac{1}{8\pi^2 n_0(x)} + \frac{1}{N} \frac{\sin[2N S_0(x) - \arcsin(x/2)]}{4\pi^3 x n_0(x)^2} \quad (64)$$

very similar to class C, but now the DOS is enhanced at the band centre because there are always two states with eigenvalue close to zero. The number of different maxima is $2N - 1$. Again this feature vanishes in the $N \rightarrow \infty$ limit, because the single additional state at $x = 0$ has measure zero for vanishing average level distance $\Delta(E) \sim 1/N$. As in the class C ensemble we have $\alpha = 1$ and the universal exponent is $\beta = 2$. Class D belongs, therefore, with class C to the unitary universality class, which determines the level repulsion even in the immediate vicinity of $E = 0$, e.g. for the second and third states away from zero.

Figures 4 and 5 give the exact DOS and the $1/N$ approximation (dashed lines) for the four ensembles.

5. Conclusions

We have computed the exact density of states for the three Wigner–Dyson ensembles as well as for classes C and D by evaluating finite-dimensional supersymmetric integrals analytically. In this way the $N \rightarrow \infty$ limit implicit in most of the previous calculations could be avoided. The exact results were then employed to test a $1/N$ expansion, which does *not* employ supersymmetric saddle points or saddle-point manifolds because it proceeds only in the bosonic sector whereas the fermionic sector is evaluated exactly. It turned out that for all ensembles only a discrete set of saddle points is important, whereas the saddle-point manifold

at $x = 0$ appearing in class C/D is not needed for an basically exact computation of the DOS everywhere in the spectrum (including $x = 0$) except the band edge. Note that this expansion may contain noninteger powers of $1/N$.

The $1/N$ expansion revealed a connection between the order α of N^{-1} multiplying the oscillatory term of the DOS and the universal short-distance exponent β of the two-level correlation function (59). In this way we obtain information about the two-point function from the one-point function. We believe that relation (59) is always fulfilled, if the spectrum is approximately ‘translational invariant’, i.e. $R_2(x_1, x_2)$ depends only on $x_2 - x_1$ (this is implicit in all calculations using an unfolding procedure). At points in the spectrum where this invariance is broken (as at $x = 0$ for class C/D) the two-point function has to be calculated itself, which can, of course, also be done with our method.

Of course, the $1/N$ expansion can also be obtained using the large- N expansion of the Hermite polynomials which appear in the exact formulae (7), (30), (37), (46), (52). But this is only easy in the unitary case, where the exact DOS can be written in terms of $H_N, H_{N\pm 1}$ only. Whereas this may be possible in the other cases as well, the calculations would become cumbersome.

The $1/N$ expansion of the DOS yields an unambiguous determination of the average level spacing $\Delta(E)$ in terms of E and N everywhere in the spectrum including $x = 0$ for all ensembles. For class C/D we conclude that the special features of the DOS at the band centre vanish in the macroscopic $N \rightarrow \infty$ limit and are of no relevance if the bandwidth is kept finite. The other Gaussian ensembles are currently under investigation.

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